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BOUNDS FOR LARGE PLASTIC DEFORMATIONS OF DYNAMICALLY LOADED CONTINUA AND STRUCTURES

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Abstract—A theorem is developed which provides bounds for maximum displacements of impulsively loaded rigid-plastic continua and structures, valid in the range of large deformations. A Lagrangian description is used. In contrast to the case of infinitesimal deformations[1], the existence of the bound is shown to be closely related to the question of stability. A simple criterion of the applicability of the theory is derived along with an equality which bounds from above permanent displacement at a chosen point of the body. The solution of an actual dynamic problem is then reduced to the determination of a statically admissible system of stresses and displacements satisfying the equations of equilibrium in the deformed configuration and violating nowhere the yield condition. Application of the theorem is given by finding estimates on moderately large deflections of beams and cylindrical shells subjected to impulsive loading.

1. INTRODUCTION

Impulsive loading theorems for elastic and rigid-plastic bodies were originally derived for geometrically linear problems[1, 2]. In extending the bounding method to large deflection problems for elastic structures Martin made use of a stability postulate in the sense of a positive increment of the potential energy away from the equilibrium state[3]. An attempt was made in [4] to give a parallel proof for a general elastic continuum starting from the convexity property of the strain energy function. However, certain terms were neglected in the expansion of that function which restricts the class of static solution. A rigorous development of the general theorem was later presented in [5].

In the early seventies Martin and Ponter[6] and one of the authors[7] proposed independently two methods of bounding from above large deflections of elastic–plastic and rigid-plastic structures. However, both approaches required the knowledge of a complete solution of the associated static problem. The method developed in [6] was subsequently generalized to cover time-dependent materials[8].

An alternative approach for studying moderately large displacements of plates was proposed in [9] where instead of a complete solution of an associated static problem, only a statically admissible solution was required. In the present paper this method is further extended to impulsively loaded rigid-plastic bodies undergoing arbitrary large deformations. An inequality is derived which bounds from above permanent displacements at a chosen point of the body in terms of an initial kinetic energy and statically admissible field of surface tractions, stresses and displacements. It has been found that the existence of an upper bound is directly related to the question of stability. The problem of dynamic stability of rigid-plastic bodies is studied in the companion paper[10] by means of the direct Liapunov method. The general theorem is illustrated by two examples involving impulsively loaded simply supported beam and cylindrical shell with ends constraint from axial motion.

2. FORMULATION OF THE DYNAMIC PROBLEM

Consider a rigid-perfectly plastic body occupying in the natural reference configuration a region $\bar{B}$ which is a subset of the three-dimensional Euclidean space E^3. Denote respectively by B and ∂B the interior and boundary of this region; the latter being a sum of sets ∂B_T and ∂B_V. The elements of the set $\bar{B}_K$ are denoted by x and called spatial coordinates. We shall study the motion of the body in the time interval $(<0, \infty)$, the element of which are denoted by t and called time variables. Assume that displacements vanish on the boundary ∂B and that the body B is in a state of initial static equilibrium with a field of body forces $\mathbf{F}_0$ and a field of surface tractions $\mathbf{T}_0$ applied to the boundary ∂B_T. A set of equations describing initial equilibrium of finitely deformed rigid-plastic body in the Lagrangian description takes the form

$$\begin{aligned} &\operatorname{div}(\mathbf{S}_0 + \nabla\mathbf{u}_0\mathbf{S}_0) = -\mathbf{F}_0 \qquad x \in B \\ &\phi(\mathbf{S}_0) \leq 0 \end{aligned} \tag{1}$$

$$\begin{aligned} &(\mathbf{S}_0 + \nabla\mathbf{u}_0\mathbf{S}_0)\mathbf{n} = \mathbf{T}_0 \qquad x \in \partial B_T \\ &\mathbf{u}_0 = 0. \end{aligned} \tag{2}$$

The existence of time-independent equilibrium state is possible provided no plastic flow takes place. Consequently, the constitutive equations do not enter to the definition of the equilibrium state. The state described by (2.1) and (2.2) constitutes a generalization of the notion of a statically admissible state to the case of large deformation problems.

Let us assume that the solution of (1) and (2) exists and is known. Apply now to the body at a chosen time certain dynamic loading. This loading consists of a suddenly applied time-independent field of body forces $\mathbf{F}$, surface tractions $\mathbf{T}$ and impulsive initial velocity field $\mathbf{v}_0$. The system of equations describing the motion of the body under the so described disturbances is given by

$$\begin{aligned} &\operatorname{div}(\mathbf{S} + \nabla\mathbf{u}\mathbf{S}) = -F_0 - \mathbf{F} + \rho_0\ddot{\mathbf{u}} \\ &\phi(\mathbf{S}) \leq 0 \qquad\qquad (x, t) \in B \times (0, \infty) \\ &\dot{\mathbf{E}} = \lambda\frac{\mathrm{d}\phi}{\mathrm{d}\mathbf{S}} \\ &\dot{\mathbf{E}} = \frac{1}{2}(\nabla\dot{\mathbf{u}} + \nabla^T\dot{\mathbf{u}} + \nabla^T\dot{\mathbf{u}}\nabla\mathbf{u} + \nabla^T\mathbf{u}\nabla\dot{\mathbf{u}}) \end{aligned} \tag{3}$$

with boundary conditions

$$\begin{aligned} &(\mathbf{S} + \nabla\mathbf{u}\mathbf{S})\mathbf{n} = \mathbf{T}_0 + \mathbf{T} \qquad (x, t) \in \partial B_T \times (0, \infty) \\ &\mathbf{u} = 0 \qquad\qquad (x, t) \in \partial B_V \times (0, \infty) \end{aligned} \tag{4}$$

and initial conditions

$$\begin{aligned} &\mathbf{u} = \mathbf{u}_0 \\ &\dot{\mathbf{u}} = \mathbf{v}_0 \qquad (x, t) \in B \times \{0\} \end{aligned} \tag{5}$$

where measures of stress $\mathbf{S}$, strain $\mathbf{E}$ and strain rate $\dot{\mathbf{E}}$ appropriate to Lagrangian description are used. In (3) λ and ρ_0 denotes respectively a scalar multiplier in the Levy-Mises flow rule and initial mass density.

3. AUXILIARY STATIC PROBLEM

Consider the same body with zero displacements on the boundary ∂B_V and loaded quasi-statically by a field of forces related to the problem of initial equilibrium and an actual dynamic problem, namely, the boundary ∂B_T is loaded by a set of surface tractions $\mathbf{T}_* + \mathbf{T}_0 + \mathbf{T}$ and the field of body forces $\mathbf{F}_* + \mathbf{F}_0 + \mathbf{F}$ is applied to the body B. The fields $\mathbf{T}_0$ and F_0 are identical to the initial loading while the fields $\mathbf{T}$ and $\mathbf{F}$ are the same as in the dynamic loading problem. The fields $\mathbf{T}_*$ and $\mathbf{F}_*$ are arbitrary.

We define a time-independent statically admissible field of stresses $\mathbf{S}_*$ displacements $\mathbf{u}_*$, surface tractions $\mathbf{T}_*$ and body forces $\mathbf{F}_*$ as one satisfying the set of equations

$$\begin{aligned} &\operatorname{div}(\mathbf{S}_* + \nabla\mathbf{u}_*\mathbf{S}_*) = -\mathbf{F}_* - \mathbf{F}_0 - \mathbf{F} \qquad x \in B \\ &\phi(\mathbf{S}_*) \leq 0 \end{aligned} \tag{6}$$

with boundary conditions

$$\begin{aligned} &(\mathbf{S}_* + \nabla \mathbf{u}_* \mathbf{S}_*)\mathbf{n} = \mathbf{T}_* + \mathbf{T}_0 + \mathbf{T} \qquad && x \in \partial B_T \\ &\mathbf{u}_* = 0 && x \in \partial B_V. \end{aligned} \tag{7}$$

We shall assume that the solution to the problem (6)–(7) exists and is known. The definition (6)–(7) is general and involves many special cases; the most simple one being $F_0 = \mathbf{F} = \mathbf{T}_0 = \mathbf{T} = 0$.

4. DERIVATION OF BOUNDING INEQUALITY

Subtracting eqns (6) with boundary conditions (7) from the eqns (3) with boundary conditions (4) and initial conditions (5) one gets

$$\begin{aligned} &\operatorname{div}(\delta\mathbf{S} + \nabla\mathbf{u}_*\delta\mathbf{S} + \nabla\delta\mathbf{u}_*\delta\mathbf{S} + \nabla\delta\mathbf{u}\mathbf{S}_*) = \mathbf{F}_* + \rho_0\delta\ddot{\mathbf{u}} \\ &\phi(\mathbf{S}_* + \delta\mathbf{S}) \le 0 \\ &\delta\dot{\mathbf{E}} = \lambda\frac{\mathrm{d}\phi}{\mathrm{d}\mathbf{S}}(\mathbf{S}_* + \delta\mathbf{S}) \\ &\delta\dot{\mathbf{E}} = \frac{1}{2}[(1 + \nabla^T\mathbf{u}_*)\nabla\delta\dot{\mathbf{u}} + \nabla^T\delta\dot{\mathbf{u}}(1 + \nabla\mathbf{u}_*) + \nabla^T\delta\mathbf{u}\nabla\delta\dot{\mathbf{u}} + \nabla^T\delta\dot{\mathbf{u}}\nabla\delta\mathbf{u}] \end{aligned} \tag{8}$$

with boundary conditions

$$\begin{aligned} &(\delta\mathbf{S} + \nabla\mathbf{u}\delta\mathbf{S} + \nabla\delta\mathbf{u}\delta\mathbf{S} + \nabla\delta\mathbf{u}\mathbf{S}_*)\mathbf{n} = -\mathbf{T}_* \\ &\delta\mathbf{u} = 0 \end{aligned} \tag{9}$$

and initial conditions

$$\begin{aligned} &\delta\mathbf{u} = \mathbf{u}_0 - \mathbf{u}_* \\ &\delta\dot{\mathbf{u}} = \mathbf{v}_0 \end{aligned} \tag{10}$$

where

$$\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_*, \quad \delta\mathbf{S} = \mathbf{S} - \mathbf{S}_* \quad \text{and} \quad \delta\mathbf{E} = \mathbf{E} - \mathbf{E}_*.$$

Multiplying the first eqn (8) by $\delta\dot{\mathbf{u}}$, integrating over B and using boundary conditions (9) we obtain

$$\begin{aligned} &\int_B \mathbf{F}_*\delta\dot{\mathbf{u}}\,\mathrm{d}(B) + \int_{\partial B_T} \mathbf{T}_*\delta\dot{\mathbf{u}}\,\mathrm{d}(\partial B) + \int_B \mathbf{S}_*(\nabla^T\delta\mathbf{u}\nabla\delta\dot{\mathbf{u}})\,\mathrm{d}(B) + \\ &\int_B \delta\mathbf{S}\delta\dot{\mathbf{E}}\,\mathrm{d}(B) + \int_B \rho_0\delta\dot{\mathbf{u}}\delta\dot{\mathbf{u}}\,\mathrm{d}(B) = 0. \end{aligned} \tag{11}$$

A time integration of (11) in the interval $\langle 0, t\rangle$ yields

$$\begin{aligned} &\int_B \mathbf{F}(\mathbf{u} - \mathbf{u}_*)\,\mathrm{d}(B) + \int_{\partial B_T} \mathbf{T}(\mathbf{u} - \mathbf{u}_*)\,\mathrm{d}(\partial B) + \frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u} - \mathbf{u}_*)\nabla(\mathbf{u} - \mathbf{u}_*)]\,\mathrm{d}(B) + \\ &\int_0^t \int_B (\mathbf{S} - \mathbf{S}_*)\dot{\mathbf{E}}\,\mathrm{d}(B)\,\mathrm{d}t + \frac{1}{2}\int_B \rho_0\dot{\mathbf{u}}\dot{\mathbf{u}}\,\mathrm{d}(B) = \int_B \mathbf{F}(\mathbf{u}_0 - \mathbf{u}_*)\,\mathrm{d}(B) + \int_{\partial B_T} \mathbf{T}(\mathbf{u}_0 - \mathbf{u}_*)\,\mathrm{d}\partial B + \\ &\frac{1}{2}\int_B \mathbf{S}_*[\nabla(\mathbf{u}_0 - \mathbf{u}_*)]\,\mathrm{d}(B) + \frac{1}{2}\int_B \rho_0\mathbf{v}_0\mathbf{v}_0\,\mathrm{d}B \end{aligned} \tag{12}$$

where initial conditions (10) have been taken care of.

The eqn (12) represents the balance of increment of energies from the stationary state. The l.h.s. of the identity (12) in the increment of the total energy of the body at an arbitrary time t while the r.h.s. denotes the initial increment of energy.

We assume the convexity-normality property of the constitutive equations to hold from which it follows that

$$(\mathbf{S}-\mathbf{S}_*)\dot{\mathbf{E}} \geq 0.$$

Denote the initial kinetic energy by $K_0=(1/2)\int \rho_0 \mathbf{v}_0 \mathbf{v}_0 \, d(B)$ and note that the kinetic energy at any time is non-negative $\int_B \rho_0 \dot{\mathbf{u}}\dot{\mathbf{u}} \, d(B) \geq 0$. Using (13), the equality (12) is now replaced by

$$\int_B \mathbf{F}_*(\mathbf{u}-\mathbf{u}_*)\,d(B)+\int_{\partial B_T} \mathbf{T}_*(\mathbf{u}-\mathbf{u}_*)\,d(\partial B)+\frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u}-\mathbf{u}_*)\nabla(\mathbf{u}-\mathbf{u}_*)]\,d(B)$$

$$\leq K_0+\frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u}_0-\mathbf{u}_*)\nabla(\mathbf{u}_0-\mathbf{u}_*)\,d(B). \tag{14}$$

Introduce now the functional $R(\mathbf{w},\mathbf{u}_*)$

$$R(\mathbf{w},\mathbf{u}_*)=\frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{w}-\mathbf{u}_*)\nabla(\mathbf{w}-\mathbf{u}_*)]\,d(B) \tag{15}$$

where the field $\mathbf{w}$ satisfies the kinematic boundary conditions for the body B. With (15), the inequality (14) can be rewritten in the form

$$\int_B \mathbf{F}_*(\mathbf{u}-\mathbf{u}_0)\,d(B)+\int_{\partial B_T} \mathbf{T}_*(\mathbf{u}-\mathbf{u}_0)\,d(\partial B) \leq$$

$$K_0+\frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u}_0-\mathbf{u}_*)\nabla(\mathbf{u}_0-\mathbf{u}_*)\,d(B)-R(\mathbf{u},\mathbf{u}_0). \tag{16}$$

If for arbitrary $\mathbf{w}$ satisfying kinematic boundary conditions holds the inequality

$$R(\mathbf{w},\mathbf{u}_*)\geq,0 \tag{17}$$

then the term R can be omitted without changing the sign of (16)

$$\int_B \mathbf{F}_*(\mathbf{u}-\mathbf{u}_0)\,d(B)+\int_{\partial B} \mathbf{T}_*(\mathbf{u}-\mathbf{u}_0)\,d(\partial B)\leq K_0+\frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u}_0-\mathbf{u}_*)\nabla(\mathbf{u}_0-\mathbf{u}_*)]\,d(B). \tag{18}$$

This is the basic inequality in the bounding theory of finitely deformed rigid-plastic bodies.

The condition (17), which is crucial for the present theorem, requires a more detailed explanation. First, we define a positive-semi definite symmetric second order tensor $\mathbf{H}$ by

$$\bigwedge_{\xi\subset\tau_1} \xi\mathbf{H}\xi\geq 0$$

where τ_1 is a three-dimensional vector space. It can be shown that a sufficient condition for R to be non-negative is that the tensor $\mathbf{S}^*$ be positive-semi definite, according to the above definition. Recall that the tensor $\mathbf{S}^*$ is positive-semi definite if the determinant of its matrix' and determinants of all its minors are non-negative. Thus, we are in the position to determine the sign of R even though the integral (15) involves an unknown actual salution w. We have shown that the sign of R can always be checked whenever the statically admissible stress field is constructed.

It would be much more difficult to state a parallel necessary condition for R to be non-negative and we have not attempted to solve this problem. A physical meaning of the functional R was further studied by one of the author (J.P.). This functional was found to be related to the geometrical stability of the structure, strictly speaking to the stability of the statically admissible stress field S* in the Liapunov sense. A detailed discussion on this problem can be found in the already mentioned companion paper, Ref. [10].

For geometrically linear problems the quadratic term on the r.h.s. of (18) and $\mathbf{u}_0$ vanish leading to

$$\int_B \mathbf{F}_* \mathbf{u}\, d(B) + \int_{\partial B} \mathbf{T}_* \mathbf{u}\, d(\partial B) \leq K_0.$$

Moreover, the functional R also becomes zero so that for infinitesimal deformations the stability problems do not come into play altogether. By taking $\mathbf{T}_* = 0$ the above inequality reduces to the familiar impulsive loading theorem due to Martin[2].†

5. PRACTICAL APPLICATION OF THE THEOREM

Return to the general case (18). Useful information about the maximum attainable displacement can be obtained by assuming either $\mathbf{T}_*$ or $\mathbf{F}_*$ equal to zero. Let us take $\mathbf{T}_* = 0$ and introduce as time-independent body force a point force $\mathbf{F}_* = \mathbf{P}_*$ acting at a particular point x_0. The corresponding statically admissible field is denoted, as before by $\mathbf{S}_*$ and $\mathbf{u}_*$. Now we obtain from (18)

$$\mathbf{P}_*[\mathbf{u}(x_0, t) - \mathbf{u}_0(x_0, t)] \leq K_0 + \frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u}_* - \mathbf{u}_0)\nabla(\mathbf{u}_* - \mathbf{u}_0)]\, d(B) \tag{19}$$

an upper bound for the maximum displacements $\mathbf{u}$ at x_0 and in the direction of the force $\mathbf{P}_*$. Let $\mathbf{P}_*\mathbf{u} > 0$ and denote by P_* the length of the vector $\mathbf{P}_*$ and by u^P and $u_0{}^P$ displacements in the direction of the vector $\mathbf{P}_*$. Under these assumptions displacement u^P is bounded from above by the known initial kinetic energy, statically admissible system $\{\mathbf{P}_*, \mathbf{S}_*, \mathbf{u}_*\}$ and initial displacements $u_0{}^P$

$$u^P(x_0, t) \leq u_0{}^P(x_0) + \frac{1}{P_*}\left[K_0 + \frac{1}{2}\int_B \mathbf{S}_*[\nabla^T(\mathbf{u}_0 - \mathbf{u}_*)\nabla(\mathbf{u}_0 - \mathbf{u}_*)]\, d(B)\right]. \tag{20}$$

Consider a plane $\{u^P, K_0\}$. For each choice of the statically admissible system, relation (20) with equality sign represents a straight line Fig. 1. The bounding curve is an envelope of the family of straight lines. This is equivalent to the optimization of the r.h.s. of (20) with respect to the load P_*.

For infinitesimal deformations we arrive again to the Martin's classical formula

$$u^P(x_0, t) \leq \frac{K_0}{P_*}. \tag{21}$$

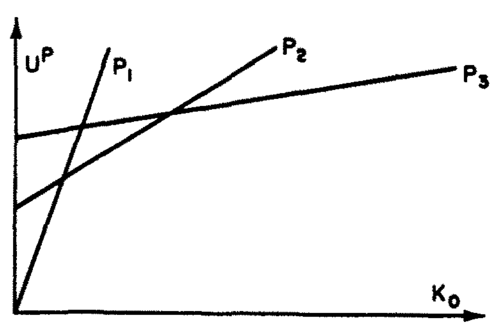

Fig. 1. Bounding curves for fixed point loading.

†For structures usually body forces are assumed to vanish $\mathbf{F}_* = 0$ while $\mathbf{T}_* \neq 0$.

6. IMPULSIVELY LOADED BEAM

Since there are no straightforward transitions from three to two-dimensional field equations, the bounding inequality for shell must be derived independently but in a similar manner as described above. However, the advantage of working with the Lagrangian description is that the final formulas can be used to write down immediately parallel expressions for moderately large deflection theory of beams, plates and shells. The applications which follow will be restricted to this theory.

Consider a simply supported beam fully restrained from axial motion and subjected to a uniform initial velocity v_0. For simplicity the axial component of the displacement vector is neglected. Geometrical parameters and coordinate systems are defined in Fig. 2.

We shall be interested in the maximum deflection at the beam mid-span as a function of the kinetic energy input K_0. A consistent set of equations of motion and geometrical relations has the form

$$M'' - (Nw')' + \mu \ddot{w} = 0 \qquad (x, t) \in (0, 2l)x(0, \infty)$$

$$N' = 0 \tag{22}$$

$$\dot{\epsilon} = w'\dot{w}' \qquad (x, t) \in (0, 2l)x(0, \infty)$$

$$\dot{K} = \dot{w}'' \tag{23}$$

where w is displacement in y direction and M and N denote respectively bending moment and axial force. Corresponding generalized strains are denoted by (K, ϵ) and their time rates by $(\dot{K}, \dot{\epsilon})$. Here prime designate differentiation with respect to the x-axis.

For a rectangular cross-section of the beam, the yield condition is described by

$$\phi \equiv \frac{|M|}{M_0} + \left(\frac{N}{N_0}\right)^2 - 1 = 0 \tag{24}$$

in which $M_0 = (\sigma_0 h^2/4)$ and $N_0 = \sigma_0 h$ denote respectively a fully plastic bending moment and axial force. The yield condition represents a closed and convex surface, Fig. 3.

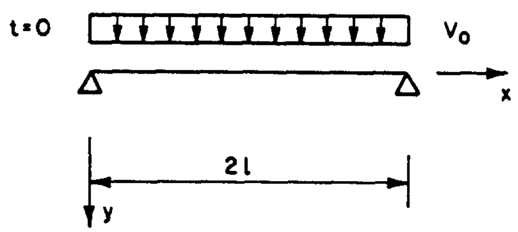

Fig. 2. Beams under a uniform initial velocity.

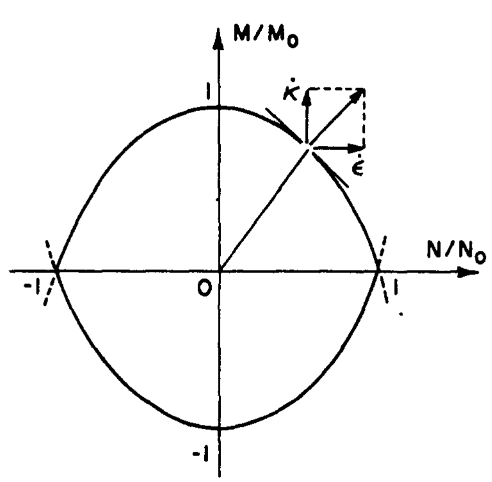

Fig. 3. Yield condition for a beam with rectangular cross-section.

The associated flow rule yields two equations

$$\dot{\epsilon} = \lambda \frac{\partial \phi}{\partial N}, \quad \dot{K} = \lambda \frac{\partial \phi}{\partial M} \qquad (x, t) \in (0, 2l)x(0, \infty). \tag{25}$$

The set of eqns (22)–(25) is supplemented by boundary conditions

$$\begin{aligned} w(0, t) &= w(2l, t) = 0 \qquad t \in (0, \infty) \\ M(0, t) &= M(2l, t) = 0 \end{aligned} \tag{26}$$

and initial conditions

$$\begin{aligned} w(x, 0) &= 0 \\ \dot{w}(x, 0) &= V_0(x) \qquad x \in \langle 0, 2l \rangle. \end{aligned} \tag{27}$$

To find an exact solution of the defined initial-boundary value problem (22)–(27) for an arbitrary initial velocity distribution would be an ambitious task. For uniform velocity a closed-form solution was given by Symonds and Mentel[11]. In the case when $K_0 l / N_0 h^2 > 1$, the formula for dimensionless permanent central deflection of the beam is

$$\frac{w_0}{h} = \sqrt{\left(\frac{K_0 l}{N_0 h^2}\right)} - \frac{1}{2} \tag{28}$$

where the initial kinetic energy is equal to

$$K_0 = \frac{1}{2} \int_0^{2l} \mu \dot{w}^2(x, 0)\, \mathrm{d}x = \mu V_0^2 l. \tag{29}$$

Prediction of the formula (28) is denoted in Fig. 5 by a dotted line.

In order to compute bounds on maximum central deflection of the beam consider an auxiliary problem in which the same structure with identical boundary conditions is loaded quasi-statically by a point load P_* applied to the beam mid-span in the y-direction.

The statically admissible field satisfies the set of equations

$$\begin{aligned} M_*'' - (N_* w_*')' - P_* \delta(l) &= 0 \qquad x \in (0, 2l) \\ N_*' &= 0 \end{aligned} \tag{30}$$

with boundary conditions

$$\begin{aligned} w_*(0) &= w_*(2l) = 0 \\ M_*(0) &= M_*(2l) = 0 \end{aligned} \tag{31}$$

and violate nowhere the yield condition (24)

$$\phi(M_*, N_*) \leqslant 0$$

where $\delta(l)$ is the Dirac function at $x = l$.

It can be proved that the general inequality (16) in the presently considered case takes the form

$$P_* w^P \leqslant K_0 + \frac{1}{2} \int_0^{2l} N_* w_*'^2\, \mathrm{d}x - R(w) \tag{32}$$

where

$$R(w) = \frac{1}{2} \int_0^{2l} N_* (w' - w_*')^2\, \mathrm{d}x.$$

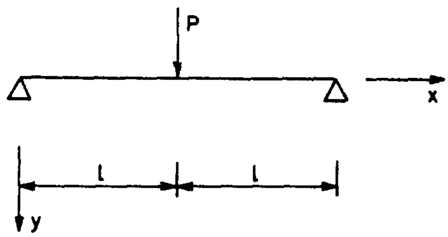

Fig. 4. Auxiliary beam problems with a concentrated force.

From the loading conditions it follows that $N_* \geq 0$ which implies $R \geq 0$ allowing (32) to be written as

$$P_* w^P \leq K_0 + \frac{1}{2}\int_0^{2l} N_* w_*'^2 \, dx. \tag{33}$$

The problem has thus been reduced to the determination of the statically admissible solution of the system (30) with conditions (31) and (32). This problem does not have a unique solution. We shall present two different solutions, each of which has a different range of applicability.

Bending solution

$$w_* = 0, \quad N_* = 0, \quad M_* = M_0 \frac{x}{l}, \quad P_* = \frac{2M_0}{l}, \qquad x \in = (0, l). \tag{35}$$

One can see by inspection that (30) and (31) are identically satisfied by (35). The state of generalized stresses is inside the yield surface except the point $x = l$ where it reaches the surface.

Substitution of (35) with (34) yields

$$w^s \leq \frac{K_0 l}{2M_0} \tag{36}$$

or in a dimensionless form

$$\frac{w^s}{n} \leq \frac{2K_0 l}{N_0 h^2}. \tag{37}$$

The bound (37) represent a straight line passing through the origin and is identical to Martin's solution for small deflection problems.

Membrane solution

$$N_* = N_0, \quad M_* = 0, \quad w_* = w_0 x, \quad P_* = 2N_0 w_0 \qquad x \in (0, l). \tag{38}$$

Now, with (38), inequality (34) reduces to

$$w_s \leq \frac{K_0}{P_*} + \frac{P_* l}{4N_0} \tag{39}$$

where w_0 and thus P_* is arbitrary. Optimizing the r.h.s. of (39) with respect to P_* in a similar way as in [3], one finally gets

$$\frac{w_0}{n} \leq \sqrt{\left(\frac{K_0 l}{N_0 h^2}\right)}. \tag{40}$$

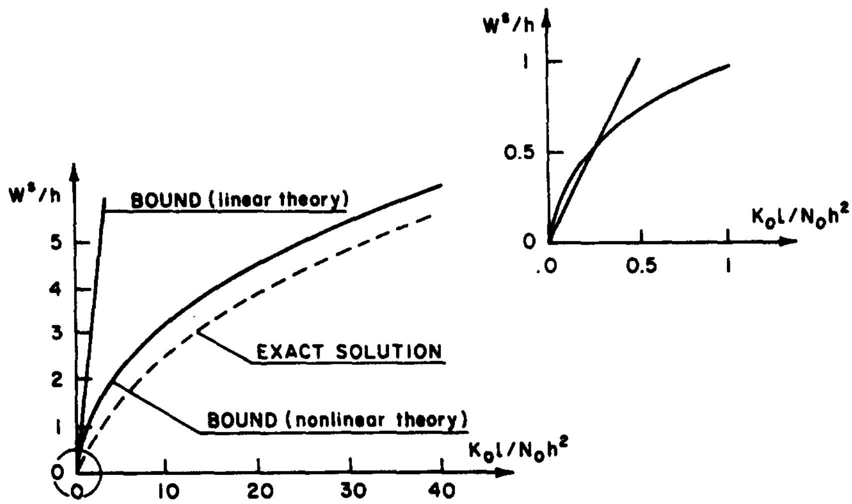

Fig. 5. Comparison of various bounds with exact solution for a beam.

Predictions of (37) and (40) are shown in Fig. 5 and can be compared with the exact solution (28). The accuracy of the bounds obtained is good and is increasing with the value of the kinetic energy K_0. An inserted figure shows regions of applicability of two alternative bounds for small values of K_0.

7. IMPULSIVELY LOADED CYLINDRICAL SHELL

As a second example consider a cylindrical shell subjected to an initial radial velocity $V_0(x)$, symmetric with respect to the center $x = l$ and giving a total initial kinetic energy input K_0. Geometrical quantities involved are defined in Fig. 6; x and θ being respectively the axial and circumferential coordinates of the shell.

The quantity we want to bound from above is the maximum central deflection of the shell. In the theory of moderately large deflections the equations of dynamics are

$$N_x' - \mu\ddot{u} = 0 \qquad -M_x'' + (N_x w')' + \frac{N_\theta}{R} - \mu\ddot{w} = 0 \qquad (x, t) \in (0, 2l) \times (0, \infty) \tag{41}$$

where u and w denote axial and radial components of the displacement vector, μ is mass density per unit area of the shell middle surface and M and N with subscripts x or θ denote as before bending moments and membrane forces. Corresponding equations for kinematics in a rate form are

$$\begin{aligned} \dot{\epsilon}_x &= \dot{u}' + w'\dot{w}' \qquad (x, t) \in (0, 2l) \times (0, \infty) \\ \dot{\epsilon}_\theta &= -\dot{w}/R \\ \dot{K}_x &= \dot{w}''. \end{aligned} \tag{42}$$

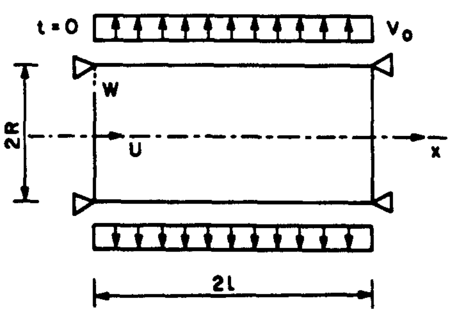

Fig. 6. A cylindrical shell under a uniform impulsive loading.

In view of rotational symmetry the component $\dot{K}_\theta$ vanishes. The vector of generalized stresses remains on or inside the yield surface $\phi = 0$. We shall use here the limited interaction yield surface, proposed by Drucker and Shield[12], Fig. 7. The associated flow rule gives three independent equations

$$\dot{\epsilon}_x = \lambda \frac{\partial \phi}{\partial N_x}, \quad \dot{\epsilon}_\theta = \lambda \frac{\partial \phi}{\partial N_\theta}, \quad \dot{K}_x = \lambda \frac{\partial \phi}{\partial M_x} \qquad (x, t) \in (0, 2l)x(0, \infty). \tag{43}$$

We assume that the shell is simply supported but prevented from the axial motion which leads to the following boundary conditions

$$\begin{aligned} &w(0, t) = w(2l, t) = 0 \qquad t \in (0, \infty) \\ &u(0, t) = u(2l, t) = 0 \\ &M_x(0, t) = M_x(2l, t) = 0. \end{aligned} \tag{44}$$

Finally, initial conditions are

$$\begin{aligned} &u(x, 0) = w(x, 0) = \dot{u}(x, 0) = 0 \qquad x \in (0, 2l) \\ &\dot{w}(x, 0) = V_0(x) \end{aligned} \tag{45}$$

A complete solution of the eqns (41)–(43) with (44) and (45) would be very difficult by fully analytical methods. In the case of a uniform initial velocity a corresponding solution was obtained by Jones[13]. We shall use this solution to plot a dimensionless actual deflection w^s/h vs dimensionless initial kinetic energy $(K_0 l/N_0 h^2)$. This relationship is shown in Fig. 9 (broken line) for short shells with a chosen characteristic parameter $c = 2l^2/Rh = 1$.

In order to compute an upper bound for w^s consider the same structure with identical boundary conditions loaded quasi-statically by a ring of forces, according to Fig. 8.

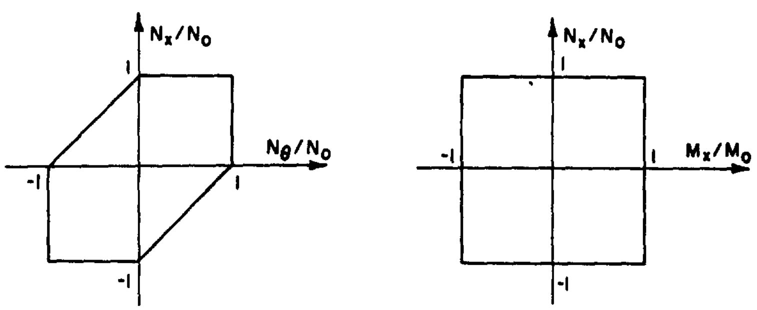

Fig. 7. Approximate yield condition for a cylindrical shell.

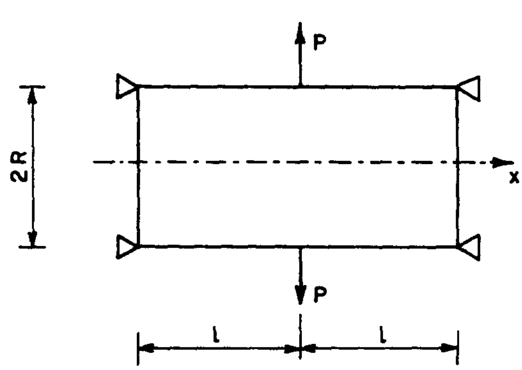

Fig. 8. A cylindrical shell statically loaded by a ring of forces.

The statically admissible solution for this problem must satisfy the equilibrium equations

$$\begin{aligned} &-M_x^{*''}+(N_x^* w^{*'})+\frac{N_\theta^*}{R}-P^*\delta(l)=0 \\ &N_x^{*'}=0 \end{aligned} \qquad x\in(0,2l) \tag{46}$$

with boundary conditions

$$w^*(0)=w^*(2l)=0$$

$$M_x^*(0)=M_x^*(2l)=0$$

and the yield condition

$$\phi(N_x^*,\quad N_\theta^*,\quad M_x^*)\leq 0. \tag{48}$$

It can be proved that in the considered case the general inequality (16) reduces to

$$P^*w^s\leq K_0+\frac{1}{2}\int_0^{2l}N_x^*(w^{*'})^2\,dx-R(w) \tag{49}$$

where

$$K_0=\frac{1}{2}\int_0^{2l}\mu\dot{w}^2(x,0)\,dxw$$

$$R(w)=\frac{1}{2}\int_0^{2l}N_x^*(w'-w_*')^2\,dx.$$

The loading conditions imply that $N_x\geq 0$, hence $R\geq 0$ and the inequality (49) is further simplified to

$$P^*w^s\leq K_0+\frac{1}{2}\int_0^{2l}N_x^*w_*'^2\,dx. \tag{50}$$

It can be noted that the statically admissible field (P^*, N_x^*, N_θ^*, M_x^*, w_*) is not unique, we shall present two solutions, each of which has a different range of applicability.

Bending solution

$$\begin{aligned} &w^*=0,\quad N_x^*=N_\theta^*=0,\quad M_x^*=M_0\frac{x}{l} \\ &P^*=\frac{2M_0}{l}. \end{aligned} \qquad x\in\langle 0,l\rangle \tag{51}$$

Substituting (51) into the inequality (50) we get the following bound:

$$w^s\leq\frac{K_0 l}{2M_0} \tag{52}$$

or in a dimensionless form

$$\frac{w^s}{h}\leq\frac{2K_0 l}{N_0 h^2}. \tag{53}$$

Membrane solution

$$w^* = -\frac{N}{2M_0R}x^2 - \left(\frac{P}{2N_0} - \frac{Nl}{N_0R}\right)x \qquad x \in \langle 0, l\rangle$$

$$N_x^* = N_0$$

$$N_\theta^* = N$$

$$M_x^* = 0 \tag{54}$$

where

$$0 \leq N \leq N_0.$$

The solution (53) combined with (50) yields

$$w^s \leq \frac{K_0 + \dfrac{N^2l^2}{3N_0R^2}}{P} + \frac{Pl}{4N_0} - \frac{Nl^2}{2N_0R}. \tag{55}$$

Optimizing the r.h.s. of (55) with respect to P one gets

$$\frac{w^s}{h} = \sqrt{\left(\frac{K_0l}{N_0h^2} + \frac{N^2l^4}{3N_0R^2h^2}\right)} - \frac{Nl^2}{2N_0Rh}, \qquad 0 \leq N \leq N_0. \tag{56}$$

The best bound would be obtained by minimizing the r.h.s. of (56) with respect to N subject to the constraint $0 \leq N \leq N_0$. Instead of doing so, we just compute the bounding curve for two different values of N. Taking $N = 0$, the formula (56) reduces to

$$\frac{w^s}{h} \leq \sqrt{\left(\frac{K_0l}{N_0h^2}\right)} \tag{57}$$

whereas for $N = N_0$ we obtain

$$\frac{w^s}{h} \leq \sqrt{\left(\frac{K_0l}{N_0h^2} + \frac{l^4}{3R^2h^2}\right)} - \frac{l^2}{2Rh} \tag{58}$$

or directly in terms of a geometrical parameter c

$$\frac{w^s}{h} = \sqrt{\left(\frac{K_0l}{N_0l^2} + \frac{c^2}{6}\right)} - \frac{c}{4}. \tag{59}$$

The present estimates compare well with Jones' solution for short shells, $c = 1$, Fig. 9. Figure 10

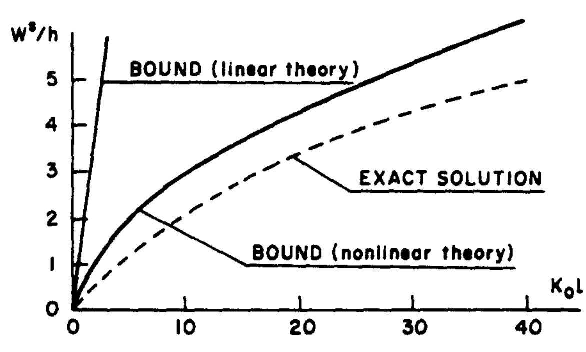

Fig. 9. Comparison of exact solution for deflection with various bounds for short shell.

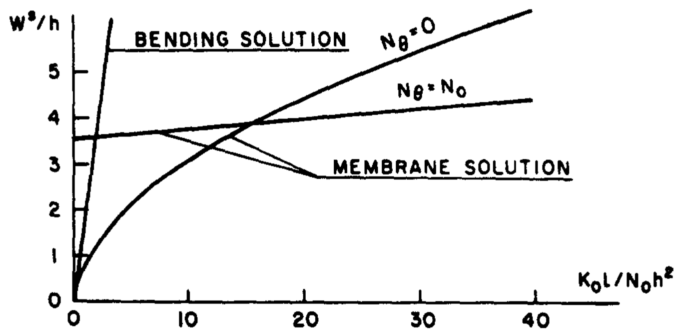

Fig. 10. Upper bound curves for a long shell.

shows the position of bounding curves for a long shell, $c = 96$. It can be seen that all three solutions (53), (57) and (59) contribute to shape of the actual bounding curve.

8. CONCLUSIONS

The existence of bounds on large displacements of impulsively loaded rigid-plastic continua and structures was shown to depend on the sign of a certain functional which is of the same form as in the stability analysis by means of the direct Liapunov method. A simple inequality is derived which bounds from above the permanent displacements in terms of the known initial kinetic energy of the body and a statically admissible system of surface tractions, stresses and displacements. The present approach takes into account dynamic effects and geometric non-linearities yet preserving the appealing simplicity of the limit analysis theorems of plasticity.

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